$b - a$. The short height h is a and the tall height H is b. Plugging these into the area formula for a vertical trapezoid, we get an area equal to $\frac{1}{2} \cdot \text{Base} \cdot (h+H) = \frac{1}{2} \cdot (b-a) \cdot (a+b)$. From basic algebra, we know that we can multiply out $(b - a)(a + b)$ to get $b^2 - a^2$. Putting this together with the $\frac{1}{2}$ from the previous expression, our calculus equation is

$$\int_a^b x = \frac{1}{2}(b^2 - a^2)$$

That takes care of the easy stuff. Let's call on our old buddy Archimedes for some assistance with $y = x^2$ and $a \leq x \leq b$. Now things are getting interesting. First consider the case where x goes from 0 to b, as shown in the first graph in Figure 5-3. The rectangular box that we inscribed around the parabolic region has a base of length b and a height of b^2. The rectangle's area is therefore $(b)(b^2) = b^3$.

Thanks to Archimedes, we know that the shaded area under the parabola is one-third of the area of the rectangle, so it's $\frac{1}{3}b^3$. But this parabolic area is more than we want. We need to subtract the part between 0 and a. That's the part shown in the second graph in Figure 5-3. It's a box of area a^3 that encloses the portion of the shaded region we need to remove in order to get the parabolic area we actually want.

As before, Archimedes lets us know that the piece to be taken away has area $\frac{1}{3}a^3$. Once we subtract that from $\frac{1}{3}b^3$, we are done. Our result is a new calculus equation that gives us the shaded region in Figure 5-4:

$$\int_a^b x^2 = \frac{1}{3}b^3 - \frac{1}{3}a^3 = \frac{1}{3}(b^3 - a^3)$$

I think we can all agree that a new pattern has been established. It has a lot in common with what we discovered in the previous chapter, but now we can compute integrals for $a \leq x \leq b$ instead of just for $0 \leq x \leq 1$. If the pattern continues, then our new and improved power rule for integrals must be

Figure 5-3.

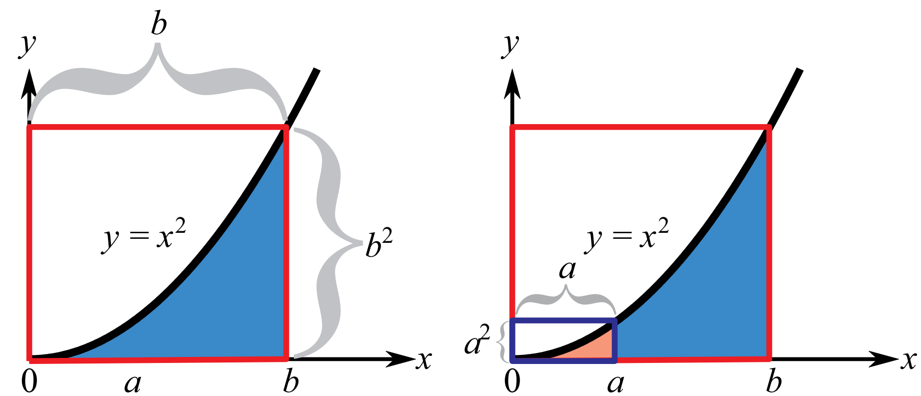
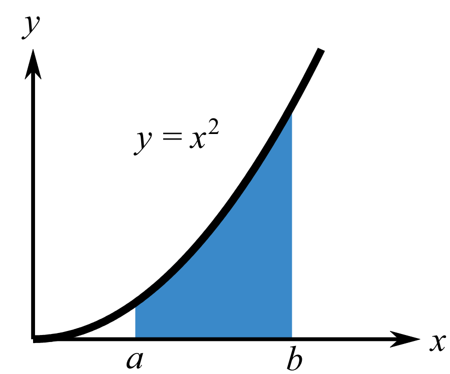


Figure 5-4.



Here are the steps:

$$
\begin{aligned}
(f(x)+g(x))' &= \lim_{\Delta x\to 0}\frac{(f(x+\Delta x)+g(x+\Delta x))-(f(x)+g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= f'(x)+g'(x)
\end{aligned}
$$

which is exactly the result we were hoping for. We have proved the addition property of derivatives.

There is also a subtraction property of derivatives. The name alone is probably enough to explain to you what it says, but let's write it down for the sake of completeness:

$$(f(x)-g(x))' = f'(x)-g'(x).$$

No surprise there.

By putting together the constant multiple rule with the addition and subtraction rules, we can find the derivative of any polynomial. It doesn't matter how many terms it has, what the powers of x are, or how big the coefficients might be. We can do it, as shown in this example:

$$
\begin{aligned}
(2x^2-7x+5)' &= (2x^2)'-(7x)'+(5)'\\
&= 2(x^2)'-7(x)'+0\\
&= 2\cdot 2x-7\cdot 1\\
&= 4x-7.
\end{aligned}
$$

(Recall that $(5)'=0$ because the derivative of any constant is zero.) Or this example:

Go forth and distribute

Taking the limit can be "distributed":

$$\lim(A+B) = \lim A + \lim B$$

values of t between a and x. In the left part of Figure 14-2, we see what $A(x)$ represents. In the right part, I've illustrated what happens when we replace x with a slightly larger number, $x + \Delta x$.

When we set up the limit of the difference quotient for $A(x)$, we have

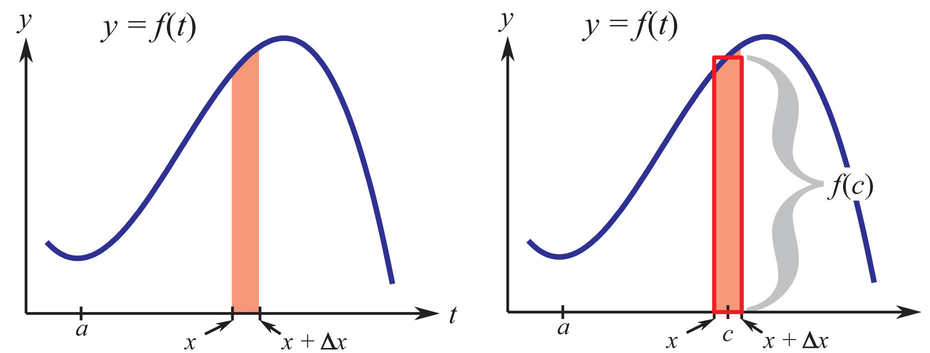
$$A'(x) = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t) - \int_a^x f(t)}{\Delta x}.$$

Now what, exactly, is equal to the difference of the two integrals? It has to be the strip of area in the left part of Figure 14-3. We can call it ΔA. In most cases, we cannot expect it to be a rectangle. However (and this is just a little bit tricky), we should be able to find a number c somewhere between x and $x + \Delta x$, as shown in the right part of Figure 14-3, so that the rectangle with height $f(c)$ and width Δx will exactly equal ΔA. The area of that rectangle is, of course, the height times the width: $f(c)\Delta x$. We can't assume that c is in the exact middle of the interval from x to $x + \Delta x$, but we don't need that. We just need its function value $f(c)$ to give the right height, after which we can write that $\Delta A = f(c)\Delta x$. (There is a mathematical theorem that guarantees the existence of the number c. It's called the *mean value theorem for integrals*.) We substitute this value for ΔA into our limit:

$$A'(x) = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t) - \int_a^x f(t)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} f(c).$$

Well, this is something different, isn't it? The limit is unlike any we've

Figure 14-3.



17 Go Forth and Multiply

Divide and Conquer, too

After a mathematical feast like the previous chapter, it will be nice to ease up somewhat and treat some gentler topics. I am going to show you the product and quotient rules for derivatives. These are extremely important rules (and the quotient rule will let us find the derivative of $\tan x$), but they are not particularly difficult to understand, even if the product rule initially threw the eminent Leibniz for a loss.

Suppose you have a function $h(x)$ that is the product of two other functions, $f(x)$ and $g(x)$. How can we find the derivative of $h(x)$? That is, how do we figure out the formula for

$$h'(x) = (f(x)g(x))'?$$

I propose to work out the formula by going all the way back to our friend the rectangle, who played such a key role in getting us started at the beginning of this book. We remember, of course, that the area A of a rectangle is the product of its length L and its width W:

$$A = LW.$$

Suppose that L is constant, but W is changing. What, then, is the change in A? Well, we already know the constant multiple rule for derivatives, the one that says $(cf(x))' = cf'(x)$. (That's the rule that says constant factors just go along for the ride, completely unaffected by taking the derivative.) Since we're assuming that L is a constant factor, we can write an equation for the rate of change (the derivative) of A:

$$A' = (LW)' = LW'.$$

where it doesn't exist, as we saw earlier), but we observe that its graph always has positive slope. Our result for the derivative of the tangent function appears to be consistent with the behavior of the graph.

Postscript

We treated the three most popular trig functions, sine, cosine, and tangent. There are three others. We mentioned secant, which is the reciprocal of cosine. The other two are cosecant, which is the reciprocal of sine ($\csc x = \frac{1}{\sin x}$), and cotangent, which is the reciprocal of tangent ($\cot x = \frac{1}{\tan x}$). While secant is useful because of its relation to tangent, the other two so-called "reciprocal" functions are mostly neglected. Still, now that you know the quotient rule, you could easily find the derivatives of all of them. If you want to test your knowledge, here is a summary of the six trig functions and their derivatives. Try to find the derivatives we didn't cover.

$f(x)$	**$f'(x)$**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$(\sec x)^2$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-(\csc x)^2$

acceleration of 32 ft/sec^2. We know that acceleration can be represented as a second derivative, but the second derivative of what?

Suppose we use the function $h(t)$ to represent the height above the ground at time t of a falling object. Then $h(0)$ is the initial height of the object at the moment we click the stopwatch and start keeping track of the height. Then $h(1)$ is the object's height after 1 unit of time, etc. What, then, is $h'(t)$? It must be the rate of change of height with respect to time. If the height is measured in feet and the time in seconds, then $h'(t)$ must be in terms of feet per second. That is, $h'(t)$ is velocity, the rate at which the object falls.

If we take the second derivative, $h''(t)$ will be the rate of change of the velocity, so it must be acceleration. That's exactly what we were looking for, but there's one more consideration before we can write an equation.

What is the sign of $h''(t)$? Because a falling object is rushing toward the ground, its height is decreasing at an ever greater rate. Both the velocity and acceleration of a falling object must be negative. We left out that little detail while talking about 32 ft/sec^2, which should really be expressed as a negative quantity. We can therefore write

$$h''(t) = -32 \text{ ft/sec}^2.$$

Thanks to our earlier work with integrals, we know how to "undo" a derivative. Let's slap an integral sign on both sides of the equation for $h''(t)$ and compute some antiderivatives:

$$\int h''(t)\,dt = \int (-32)\,dt$$
$$h'(t) = -32t + C_1.$$

Does that look okay to you? The antiderivative of $h''(t)$ has to be $h'(t)$—we just take one prime away—and the antiderivative of a constant like –32 has to be $-32t$. Then there's the arbitrary constant (we mustn't forget that), which I wrote with a subscript of 1. Why?

What we've discovered is that there's only one way for the exponential function to be represented as an infinite-degree polynomial, and that's if the nth coefficient is a_0 divided by n! If you look at our previous results, you'll see that this fits. The formula for a_n is given by

$$a_n = \frac{1}{n!}$$

We can now write our infinite-degree polynomial for e^x. Since $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$, we have

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots.$$

What happens if we take the derivative of this expression? We're supposed to get the same thing back. Let's try it:

$$\begin{aligned}
(e^x)' &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right)' \\
e^x &= 0 + 1 + \frac{1}{2} \cdot 2x + \frac{1}{6} \cdot 3x^2 + \frac{1}{24} \cdot 4x^3 + \frac{1}{120} \cdot 5x^4 + \cdots \\
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots.
\end{aligned}$$

Yes, it worked perfectly.

The preferred mathematical name for infinite-degree polynomials is *power series*. Not all functions have power series as nice as the one for the exponential function, but power series have broad applicability in higher math.

Just a handful

Remember how, at the beginning of the chapter, I mentioned the difficulty of computing $e^{1/2}$ without the aid of a calculator or printed table of values? Let's take another look at that problem. With the aid of our power series, we see that